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LETTER TO THE EDITOR

Two remarks on a paper by Sani *et al.*

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SUMMARY

In Sani *et al.* (*Int. J. Numer. Meth. Fluids* 2006; **50**:673–682), the authors claim to provide a proof of well-posedness for certain formulations of the Navier–Stokes equations for incompressible flow. We consider the proof of their main Theorem 1 incomplete, and point out some inconsistencies in the above paper. Copyright © 2008 John Wiley & Sons, Ltd.

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1. INTRODUCTION

The paper [1] by Sani *et al.* represents another iteration in a long-running argument over the well-posedness of certain formulations of the Navier–Stokes equations for incompressible flow. This argument arises most often, but not exclusively [2], in the context of formulations that replace the equation of mass conservation for incompressible flow with an equation for the pressure. Because we will cite this paper frequently, we will refer to [1] as S06 below. For the convenience of the reader, we will start by restating the equations in question, and we will follow with two critical remarks regarding the material in S06. The paper ends with a brief conclusion.

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2. THE EQUATIONS

S06 examines alternative formulations for the Stokes equations,

$$\frac{\partial \mathbf{u}}{\partial t}+\nabla p-\nu\nabla^2\mathbf{u}=\mathbf{f} \tag{1}$$

$$\nabla\cdot\mathbf{u}=0 \tag{2}$$

where $\mathbf{x}\in\Omega$ is the vector of Cartesian coordinates, $\mathbf{u}$ is the velocity vector, p the pressure divided by the constant density ρ of the fluid, ν the kinematic viscosity, and Ω is the spatial domain of interest. For well-posedness, these differential equations need to be complemented by initial and boundary conditions, and we will assume that these are provided by giving an initial condition $\mathbf{u}_0$ which is C^∞ in all of Ω,

$$\mathbf{u}(\mathbf{x},t_0)=\mathbf{u}_0(\mathbf{x}) \quad \text{with } \nabla\cdot\mathbf{u}_0=0,\ \ \mathbf{x}\in\Omega \tag{3}$$

and boundary conditions

$$\mathbf{u}(\mathbf{x},t)=\mathbf{u}_\Gamma(\mathbf{x},t), \quad \mathbf{x}\in\Gamma \tag{4}$$

satisfying the global mass conservation constraint

$$\oint_\Gamma \boldsymbol{\eta}\cdot\mathbf{u}_\Gamma\,\mathrm{d}\sigma=0 \tag{5}$$

on the boundary $\Gamma=\partial\Omega$ of the domain, where $\boldsymbol{\eta}$ is a vector normal to the boundary.

For reasons of numerical expedience discussed in more detail elsewhere [2], many computational algorithms do not directly solve the system of equations given above. Rather than solving the continuity equation (2), they replace it by a pressure Poisson equation, obtained by taking the divergence of (1) and using (2). This leads to

$$\nabla^2 p=\nabla\cdot f \tag{6}$$

an equation commonly known as the pressure Poisson equation, but termed the 'simplified pressure Poisson equation' (SPPE) in S06. Alternatively, S06 also introduces a so-called 'consistent pressure Poisson equation' (CPPE) which is obtained by keeping the velocity-divergence term,

$$\nabla^2 p-\nu\nabla\cdot(\nabla^2\mathbf{u})=\nabla\cdot f \tag{7}$$

Thus, we can now consider two alternative formulations of the Stokes problem: One is the system based on the SPPE, (1), (3)–(6), the other one is based on the CPPE and given by (1), (3)–(5), (7). For convenience, we will refer to these systems as S-SPPE for the former and S-CPPE for the latter.

An important question is the one of whether or not, and under what conditions one or both of these systems of equations are equivalent to the original system (1)–(5). S06 attempts to answer this question, but we find the answers given there in part inconsistent (our First Remark), and in part incorrect (our Second Remark). Our criticism is laid out in the following section.

We emphasize that this topic has been discussed in the literature extensively. The most recent example we are aware of, in addition to S06 and [2], is a paper by Johnston and Liu [3] that agrees with some of the conclusions in S06. We will not discuss any of this material here, but we note

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that our review paper [2] does contain an extensive bibliography along with some rigorous results on the subject.

3. TWO REMARKS

3.1. First remark—an inconsistency

Clearly, the main intended thrust of S06 lies in a proof that S-CPPE represents a well-posed mathematical problem, which is equivalent to the original Stokes equations (1)–(5). We will return to this claim in our Second Remark, but for now we note that S06 does not claim, nor attempt to prove, that S-SPPE is equivalent to (1)–(5). Indeed, both at the beginning of their discussion section (Section 3 in S06) as well as at the very end of their paper, the authors *seem* to imply that for S-SPPE to be well-posed, it needs to be complemented by a boundary condition of the form

$$\nabla \cdot \mathbf{u}(\mathbf{x}, t) = 0, \quad \mathbf{x} \in \Gamma \tag{8}$$

i.e. we need to explicitly enforce the continuity equation at the boundary Γ of the domain. To wit, in their discussion section, the authors comment that 'the BC $\nabla \cdot u|_\Gamma$ is essential…', and at the very end of their paper they say that '…we believe and assert that a very important aspect of incompressible flow is the requirement that $\nabla \cdot u = 0$ on $\Omega + \Gamma$ and for all $t \geqslant 0$.' We agree with both of these statements.

However, we notice that the above differs from earlier pronouncements by some of the same authors (see, e.g. [4–6]). In these earlier publications, the authors had insisted that S-SPPE becomes well-posed and is equivalent to (1)–(5) when augmented by a boundary condition obtained by writing down the normal projection of (1) at the boundary. Since this approach results in a condition for the normal pressure gradient, the resulting equation is often referred to as 'the Neumann boundary condition for pressure'. For the case of a solid wall, we would have

$$\frac{\partial p}{\partial \eta}(\mathbf{x}, t) = \boldsymbol{\eta} \cdot (\nu \nabla^2 \mathbf{u}(\mathbf{x}, t) + \mathbf{f}), \quad \mathbf{x} \in \Gamma \tag{9}$$

Thus, the assertion was that (1), (3)–(6), (9) represent a system of equations that is equivalent to (1)–(5) with $\mathbf{u}_\Gamma \equiv 0$. It is this claim that is presented as the 'equivalence theorem (assertion)' in [6]. Yet, this assertion is falsified by the proof of our Theorem 1 in [2] which states that the system (1), (3)–(6), (9) is ill-posed and has an infinite number of solutions. We emphasize that this fact is not subject to debate: [2] contains both a formal proof as well as a practical demonstration of its truth. By constructing an exact solution to a 'Stokes problem' that satisfies S-SPPE with the boundary condition (9), but that is not a solution to the Stokes problem (1)–(5), we have demonstrated that S-SPPE plus (9) does not constitute a proper formulation of the Stokes problem. Indeed, the passages from S06 we have quoted may be read to indicate that the authors of S06, too, are willing to withdraw their erroneous 'equivalence theorem (assertion)'.

However, we find the abstract of S06 incongruent both with the above reading and with the rest of S06. This abstract begins by referring back to the 'equivalence theorem (assertion)', announcing that S06 will provide a proof of this theorem 'for at least some situations'. While it is not entirely clear from the abstract of S06 which of the above formulations is being referred to (the authors were careful to distinguish between the CPPE and the SPPE elsewhere in their paper, but they simply refer to the PPE in their abstract), going back to [6] there is no doubt that their original

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'equivalence theorem' dealt with the Navier–Stokes equivalent of the system S-SPPE with an additional Neumann boundary condition. In contrast, S06 focuses entirely on S-CPPE, and thus does not contain any attempt to prove an 'equivalence theorem' for S-SPPE.

As a minor point, we also note that in the introduction of S06, a statement by the present author on the ill-posedness of S-SPPE [7] is cited as an example of 'confusion in the literature on the pressure BC issue'. We find this remark puzzling, given that, despite the declaration in S06 that 'This statement [of the present author] is proven herein to be absolutely incorrect', no such proof is given nor attempted anywhere in S06. We do agree with the authors of S06 that there may be some confusion associated with this topic, but we believe that it is not ours.

3.2. Second remark—a non-sequitur

We now come back to consider the main result of S06, which is an attempt to prove that S-CPPE is well-posed and represents an equivalent formulation of the Stokes problem (1)–(5). The following observations are pertinent:

- While the mathematics of the system S-SPPE is well understood, this is not at all the case for S-CPPE. Indeed, the authors in S06 note correctly that 'there were no rigorous mathematical analyses for this formulation...'. We agree, but we note that no such analysis is presented by these authors either. Specifically, we emphasize that the system S-CPPE is fundamentally different, and of higher order than S-SPPE. What may appear like an innocuous addition of a term to S-SPPE does in fact result in a system of partial differential equations (PDEs) that is quite different from S-SPPE. Any serious analysis of this system would have to start by describing the nature of solutions to this system, and determining what boundary conditions are needed to make the solution unique. While it is near trivial to show that S-SPPE will have unique solutions when Dirichlet conditions for both the velocities and the pressure are given, it would require a very serious effort to understand what boundary conditions, if any, would generate unique solutions for the system S-CPPE. In this context, and as a warning, it may be instructive to consider the case of the steady Stokes problem. It is well known [3] and easy to see that the steady-state version of the set of PDEs in S-CPPE,

$$\nabla p - v\nabla^2 \mathbf{u} = \mathbf{f} \tag{10}$$

$$\nabla^2 p - v\nabla \cdot (\nabla^2 \mathbf{u}) = \nabla \cdot \mathbf{f} \tag{11}$$

  represents an ill-posed problem for *any* choice of boundary conditions, simply because (11) is just the divergence of (10), meaning that we do not have a sufficient number of independent PDEs to determine a unique solution.
- While the authors do not perform the analysis of S-CPPE we found it necessary; they attempt to circumvent this need by trying instead to prove that S-CPPE is equivalent to (1)–(5) in the sense that any solution of one is also a solution of the other and *vice versa*. If this strategy was successful, they could refer to results on the well-posedness of (1)–(5) and thus establish well-posedness of S-CPPE.

  Unfortunately, their proof of equivalence does not hold up to scrutiny. Examining the proof we find that it is attached to the fact that any solution to S-CPPE will satisfy the equation

$$\frac{\partial}{\partial t}(\nabla \cdot \mathbf{u}) = 0 \tag{12}$$

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as can easily be seen by substituting CPPE into the divergence of (1). The authors then go on to conclude that, because of the divergence-free initial condition (3), the solution will remain divergence free for all times. We note in passing that similar 'proofs' have appeared in the literature before [3, 6].

The above attempt at a proof is missing a crucial piece: The authors have not established that the divergence of any solution $\mathbf{u}$ of S-CPPE must be continuous in time at $t=0$. Notice that the existence of the left-hand side of (12) is only guaranteed for $t>0$. For their proof to be complete the authors therefore still need to show existence of the time derivative at $t=0$. It is clear that without this essential element, the conclusion does not follow, and S06 falls considerably short of its stated goal.

4. CONCLUSIONS

The work in S06 and similar papers is driven by the desire to develop an exact factorization of the Navier–Stokes equations such that the equations for velocity and pressure at each time step are decoupled as much as possible. We continue to believe that this program can succeed only to a limited extent. For the so-called influence matrix methods (see, e.g. [8]), we almost succeed, but at the price of a global coupling of the boundary pressure to the velocities everywhere on the boundary. Alternatively, we may choose a fractional step method, which perfectly decouples the equations, but at the cost of a factorization error.

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